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LETTER TO THE EDITOR

Dynamical 'strangeness' at the edge of chaos

A Politi†§ and R Badii‡

† Istituto Nazionale di Ottica, Firenze, Italy
‡ Paul Scherrer Institute, 5232 Villigen, Switzerland

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Abstract. A recently introduced class of dynamical systems, the generalized shifts, is shown to exhibit both topological chaos, while being metrically ordered, and a phase transition in the framework of the thermodynamic formalism when this is applied to its entropic properties. Therefore, it provides a further example of complex behaviour emerging at the border between order and chaos. Its anomalous dynamical properties are the result of a strong nonhyperbolicity which, in turn, is a manifestation of uncomputability in generalized shifts.

Chaos and, particularly, metric chaos, defined by the positivity of at least one Lyapunov exponent [1], has long been recognized as a source of unpredictability. No matter how small the uncertainty of the present state of the system is, predictions about the future evolution are affected by exponentially increasing errors. This form of randomness, although the commonest one, is, however, not the only source of unpredictability in low-dimensional deterministic nonlinear systems. Examples can be constructed [2] in which any trajectory is asymptotically attracted by one of two (or more) invariant sets, the basins of attraction of which are so intricately intertwined that arbitrarily high precision is required to predict the ultimate fate of the system (riddled basins [3]). This phenomenon is reminiscent of the undecidability problem occurring in the theory of computation [4]: in fact, no algorithm presumably exists which is able to assess in a finite number of steps the membership of a set X_0 of initial conditions to one of the basins, no matter how small X_0 is. It may be noted that a finite computation is, instead, generally sufficient to solve the analogous problem in a system with one attractor and one repellor.

A closer relationship with computation theory is exhibited by the generalized shifts (GSs), proposed by Moore [5] as a 'bridge' between dynamical systems and Turing machines. These shifts are, indeed, recognizable as dynamical systems, provided that an appropriate coding is used to translate their rules into the action of a two-dimensional map (see later). The parallel with Turing machines shows that there is conceivably no general algorithm from which the asymptotic behaviour of a generic orbit can be deduced. Because of the vanishing of the Lyapunov exponents in the example of [5], it is not possible to decide *a priori* whether one observes chaotic motion or just a very long periodic orbit. In fact, the complexity in the evolution of a GS follows from the continuous exchange of the stable and unstable directions. Although similar to that of nonhyperbolic maps [6], the situation here is worsened by the lack of a global horseshoe mechanism, as confirmed by

§ Also with INFN, senzione di Firenze, Italy.

the extreme rarefaction of the set of periodic orbits which, moreover, contains nearly no short cycles [5, 7].

In the present work, we study the specific GS introduced in [5] because of its intriguing properties and show that it is a marginally chaotic system, characterized by the positivity of both the topological entropy K_0 and the generalized expansion rates (entropies) $L(q)$ for $q < 1$, and by non-analytic behaviour of the 'free-energy' $L(q)$ around $q = 2$, which is indicative of a first-order phase transition in the thermodynamic formalism for dynamical systems [8–10]. Therefore, this system constitutes a further example of complex behaviour at the border between order and chaos, which may be added to the well known phenomena of period-doubling [11], intermittency [12], strange non-chaotic attractors [13], and some product systems [14]. While the largest Lyapunov exponent is zero in all of them, the time correlation function decays as a power law in the former three and exponentially in the latter and in Moore's GS (at variance with the GS, however, the system of [14] has zero topological entropy).

For the reader's convenience, we briefly recall the definition of the generalized shift [5, 7]. Given an infinite sequence $\mathcal{S} = \ldots s_{i-1}\hat{s}_i s_{i+1} \ldots$ of symbols $s_i \in \{0, b-1\}$, with a distinct position (marked by the pointer), an updating rule replaces the word w of length $2r+1$ centred at the reference symbol by a new word w' and displaces the pointer by one position. Both the new word w' and the direction of the shift are determined by w. Table 1 illustrates the binary rule ($b = 2$) with range $r = 1$ introduced in [5] (using a different coding).

Any GS is conjugated to a piecewise linear map of the plane [7]: in fact, the two semi-infinite sequences at the sides of the pointer define the coordinates (x, y) of a point in the unit square through the binary expansions $x = \sum_{i=-1}^{-\infty} 2^i s_i$ and $y = \sum_{i=0}^{\infty} 2^{-i} s_i$ (the right sequence including, by convention, the symbol under the pointer). Hence, a right shift yields a contraction along the x-direction and an expansion along the y-direction, both by a factor two. The opposite occurs for the left shift. Furthermore, the substitutions correspond to translations of rectangles inside the unit square: as long as the dynamics is invertible, it is also area-preserving as can be easily verified from figure 1.

The behaviour of the generalized shift can be profitably studied by mapping the movement of the pointer to a symbolic sequence $\mathcal{T} = \ldots t_{i-1}t_i t_{i+1}$, in which $t_i = 0$ (1) corresponds to a left (right) displacement. This coding is a full representation of the GS since it corresponds to the symbolic dynamics of the GS-map with a binary partition defined by $y = 1/2$ in the image square (on the right in figure 1). In fact, if $y < 1/2$ ($> 1/2$), the pointer is positioned on a 0 (1) in the image triple w' (see table 1). Moreover, this partition is also generating because each triple w' univocally identifies one of the rectangles in the square.

An association between GSs and Turing machines [4] has been made in [5, 7], whereby the substitutions are assimilated to a change in the machine's internal state and the pointer to the tape head. Therefore, the long-time behaviour of a GS is unpredictable, not just because of sensitivity to the initial conditions but, even more, because of the undecidability of any general question concerning the state of the machine. For example, no algorithm can establish whether the pointer will ever reach any preassigned position for generic initial conditions. Of course, this remark holds only if it is indeed impossible to reproduce the specific GS with an automaton belonging to a lower class than a Turing machine. Although this analogy does not render GSs more interesting than shift dynamical systems (as shown above, a GS does correspond to an 'ordinary' subshift dynamics), it helps revealing that the latter class of systems can exhibit a higher degree of unpredictability than that due to deterministic chaos. As usual in the theory of computation, however, it is very hard to prove that a specific model is intimately unpredictable.

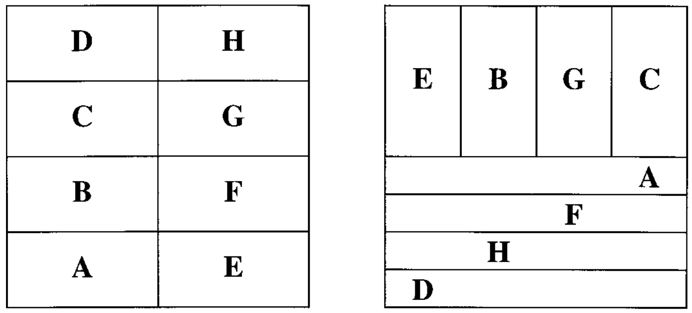

Figure 1. Illustration of the action of the GS-map on the unit square. An eight-element partition (left) and its image (right) are indicated.

Table 1. Substitution rules $w \to w'$ for Moore's generalized shift. The letters from A to H label the eight phase-space partition elements which correspond to w in the associated two-dimensional map.

Label	w	w'
A	$0\hat{0}0$	$\hat{0}11$
B	$0\hat{0}1$	$10\hat{1}$
C	$0\hat{1}0$	$11\hat{1}$
D	$0\hat{1}1$	$\hat{0}00$
E	$1\hat{0}0$	$00\hat{1}$
F	$1\hat{0}1$	$\hat{0}10$
G	$1\hat{1}0$	$01\hat{1}$
H	$1\hat{1}1$	$\hat{0}01$

A direct numerical simulation for several distinct choices of the initial condition and involving more than 10^{10} iterates suggests the existence of a single ergodic component covering almost uniformly the whole unit square (up to measure-zero sets). By tiling the phase space with square cells of linear size 2^{-8}, the ratio between the probability of the heaviest and the lightest cell was estimated to be about 1.7, without significant changes upon increase of either the spatial resolution or the statistics (number of iterations). Furthermore, the power spectrum of the shifts' sequence $\mathcal{T}$ appears to be continuous, with a few prominent broadened peaks. In other words, the direct numerical analysis indicates that this model is not qualitatively different from a typical chaotic system with exponentially decaying correlations. The only exception to this picture is represented by strong deviations from a uniform phase-space filling, which are observed at the spatial scale 2^{-8} even after 10^7 iterations: this phenomenon represents the first hint of weak chaos.

With the knowledge of the symbolic dynamics, we could identify all the N_n legal subsequences T of the signal $\mathcal{T} = \dots t_{i-1} t_i t_{i+1}$ with length $|T| = n$, for $n = 1, 2, \dots, 80$. As a result, the topological entropy $K_0 = \lim_{n\to\infty} (\ln N_n)/n$ has been estimated to be approximately equal to 0.14 [10], although perfect convergence for $n \gg 1$ has not been achieved by this direct method. These finite-size estimates, however, exclude the possibility that $K_0 = 0$, a value that might be suggested by the vanishing of the largest Lyapunov exponent [5]. Accordingly, this model can be interpreted as a further example of a system at the 'edge of chaos', with the additional remarkable feature of the positivity of the topological entropy. Moreover, although the vertical and horizontal directions are invariant (no rotation is present), the area-preservation constraint implies that one of them expands when the other contracts, so that both can, in principle, contribute to the entropy.

In order to clarify these mechanisms, we have studied the periodic orbits $T^\infty = (t_1 t_2 \ldots t_p)^\infty$ of the system (where p is the period and the superscript indicates infinite repetition of the basic sequence). The orbits identified in [5] are the two fixed points 0^∞ and 1^∞ (i.e. arbitrarily many left and right shifts, respectively), the two cycles $T_7 = (0000111)^\infty$ and $T_7' = (0001101)^\infty$, and a family of period-$(15+n)$ orbits with $n \geqslant 1$. This is a very thin set which gives no clue about the possible positivity of the topological entropy. Here, we show that this set can be greatly enlarged, by proving that any combination of the words $T_1 = 1$ and $T_{16} = 1001011010010111$ corresponds to an admissible trajectory of the dynamical system. Indeed, the spatial sequence $S = 1\hat{1}0$ is shifted to the right (i.e. the GS 'emits' the symbol $t = T_1 = 1$) if a 0 is present to its right. If, instead, S is followed by 10, the GS reproduces it, displaced by two steps to the right, after 16 iterations during which T_{16} is emitted. Therefore, provided that no two consecutive 1s are present to the right of S, the above procedure can be repeated infinitely many times, thus giving rise to any arbitrary concatenation of T_1 and T_{16}. Notice that the family of period-$(15+n)$ orbits found in [5] corresponds to the particular case $(T_1^{n-1} T_{16})^\infty$.

The generalized entropy function H_q associated with the ensemble $\mathcal{L}_{uv}$ of all possible concatenations of two words u and v, of lengths n_u and n_v, is the solution of the equation [15]

$$p_u^q \, \mathrm{e}^{(q-1)H_q n_u} + p_v^q \, \mathrm{e}^{(q-1)H_q n_v} = 1 \tag{1}$$

where p_u and p_v are the respective probabilities. In the present case, $n_u = 1$ and $n_v = 16$, while p_u and p_v can be identified with the inverse multipliers $1/2$ and $1/4$, respectively, of the corresponding periodic orbits. For $q = 0$, we obtain $H_0 = 0.131\,05\ldots$ (independently of the p_is) which, although a lower bound to the topological entropy K_0 of the GS, is rather close to the estimate for K_0 found from the direct enumeration of all possible sequences.

A more detailed, although less rigorous, analysis of the statistical properties of this GS can be performed by approximating its dynamics with Markov processes of increasing order. Because of the piecewise linearity of the map, this can be done by partitioning the unit square into the largest rectangles $R_i(n)$ such that their nth image is still a single (i.e. unsplit) rectangle. An example of the partition obtained for $n = 4$ is reported in figure 2.

The n-step transition probabilities $p_{ij}^{(n)} = P(R_j|R_i)$ can be approximated by the area of $\boldsymbol{F}^n(R_i) \cap R_j$, where $\boldsymbol{F}^n$ is the nth iterate of the GS map $\boldsymbol{F}$, because of the 'experimentally' observed uniformity of the invariant measure. Accordingly, one defines the transition matrix

$$M_{ij}^\alpha(q; n) = p_{ij}^{(n)} \mu_i^\alpha(n)^{1-q} \tag{2}$$

where $\mu_i^\alpha(n)$ is the expansion factor of the ith rectangle along the αth direction ($\alpha = x$ or y) in n steps ($\mu_i^\alpha(n) = 2^k$, with $1 \leqslant k \leqslant n$). The corresponding generalized Lyapunov exponents $L_\alpha(q; n)$ are then given by

$$L_\alpha(q; n) = \frac{\ln \Lambda_\alpha(q; n)}{1 - q} \tag{3}$$

where $\Lambda_\alpha(q; n)$ is the largest eigenvalue of $\boldsymbol{M}^\alpha(q; n)$. Since the map is area preserving, $\mu_i^x = 1/\mu_i^y$ and $L_x(q; n) = L_y(2 - q; n)$.

An equivalent representation of the statistical properties of the GS is provided by the Legendre transforms of $L_\alpha(q; n)$: i.e. by the spectra $h_\alpha(\lambda)$ of Lyapunov exponents. Since these functions lend themselves to a more direct interpretation, we shall comment our results with reference to their properties. Several features are worth being commented upon. First, the supports of both spectra $h_\alpha(\lambda)$ extend to negative λ (see figure 3), thus confirming the

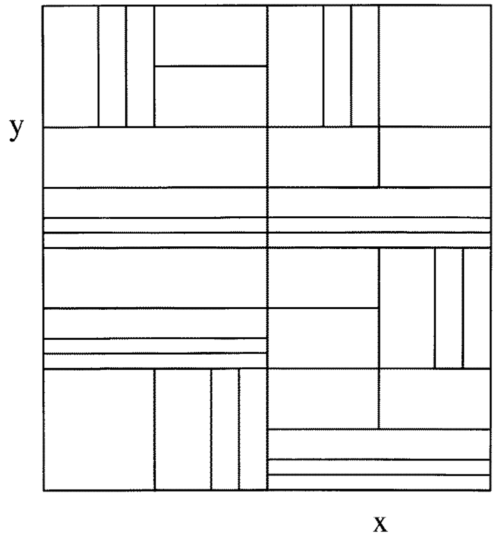

Figure 2. Order-four partition used in the construction of the Markov approximation of the GS dynamics.

strong nonhyperbolicity of this model†. Second, the difference between the two spectra far from the region $\lambda \approx 0$ reveals a clear asymmetry between left- and right-shifting GS operations. Notwithstanding this, the Lyapunov exponent is remarkably equal to 0 (its value actually laying below the numerical accuracy, independently of the order of the Markov process).

From the two Lyapunov spectra, one can also infer the shape of the overall spectrum $g(\kappa)$ of the generalized entropies (the Legendre transform of the function K_q). By definition, only the expanding directions contribute to the entropies. For each trajectory with expansions occurring along the x-direction there is an equally strong contraction along y and *vice versa*: hence, $g(\kappa)$ is obtained as the maximum between $h_x(\lambda)$ and $h_y(\lambda)$ for all positive $\lambda = \kappa$s. We find that, as a result of the asymmetry, the x-direction alone determines the spectrum of entropies.

With reference to the curve $g(\kappa) = h_x(\lambda)|_{\lambda=\kappa>0}$, we observe that the topological entropy K_0 (i.e. the maximum of the spectrum) already converges quite well for n as small as 4 to the value $K_0 = 0.131(1)$, which is extremely close to the lower bound determined above from equation (1). This suggests that the family $\mathcal{L}_{T_1 T_{16}}$ of sequences obtained by concatenating T_1 and T_{16} captures almost the full richness of the GS trajectories. Even if this bound were saturated, however, we could still not conclude that the dynamical complexity of the GS is exhausted by the classification of the sequences in $\mathcal{L}_{T_1 T_{16}}$. In fact, the trajectories that give the leading contribution to the topological entropy are indeed nowhere dense in phase space. Even worse, the closure of the set of all such orbits is made of points (x, y) whose binary expansions have no two consecutive 1s (on either side of the substitution domain of the GS). Hence, its fractal dimension $D = 2\ln[(1+\sqrt{5})/2]/\ln 2 = 1.388\ldots$ is definitely smaller than the dimension of the support of the invariant measure (apparently equal to two).

† The maximum and minimum local Lyapunov exponents, equal to $\pm\log 2$, are nothing but the expansion rates of the two fixed points.

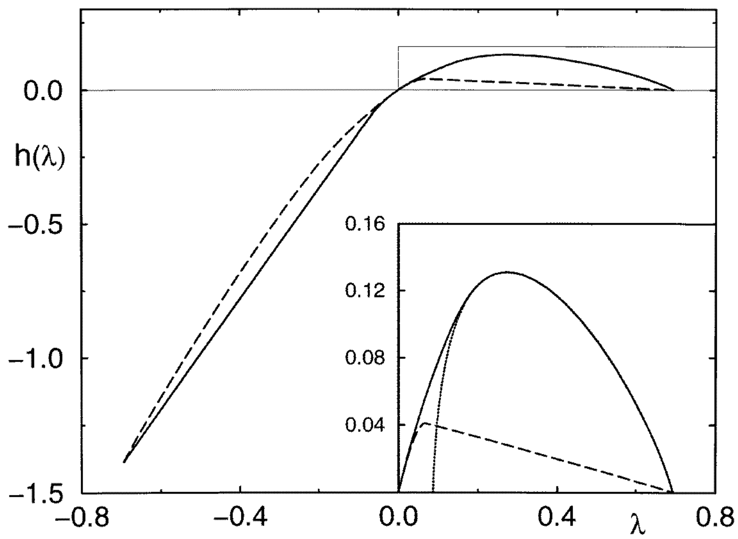

Figure 3. Spectra $h_\alpha(\lambda)$ of Lyapunov exponents against λ, at the approximation order $n = 20$, where $\alpha = x, y$ (full and dashed curves, respectively) indicates the direction in phase space. The inlet shows a vertical expansion of the curves $h_\alpha(\lambda)$ for positive λ, superimposed on the analogous curve $\tilde{h}(\lambda)$ (dotted curve) obtained from all concatenations of the two sequences T_1 and T_{16}.

The accuracy in the approximation of the GS dynamics obtained from the set $\mathcal{L}_{T_1T_{16}}$ can be evaluated by comparing $h_x(\lambda; n)$ with the spectrum $\tilde{h}(\lambda)$ given by $\mathcal{L}_{T_1T_{16}}$. As seen in figure 3, the agreement is excellent for $\lambda \geqslant 0.15$, while $\tilde{h}(\lambda)$ fails completely to reproduce the real behaviour in the vicinity of $\lambda = 0$. This is clearly due to the hyperbolic structure of $\mathcal{L}_{T_1T_{16}}$ whose minimum Lyapunov exponent is that of the period-16 orbit, which is strictly positive.

We have further tried to construct an analogous approximation for the orbits with a left pointer shift. The curve $h_y(\lambda)$ indeed shows that they also have a positive topological entropy, although smaller than the overall entropy. Unfortunately, the three possible candidate orbits found in [5–7] (one fixed point and two period-7 cycles) cannot be concatenated with one another. A computer-aided search has revealed the existence of longer periodic orbits which, however, again cannot be concatenated. We just mention the shortest one, of period 128, which corresponds to the spatial configuration $(01)^\infty 10100100011(0001)^\infty$ with a shift by four sites to the left in a whole period.

Finally, notice the linear shape of h_x (h_y) for $\lambda < -0.065$ ($\lambda > 0.065$), suggestive of a phase transition, which could indicate the absence of periodic orbits with contraction rates close to that of the fixed point (our search has revealed only cycles with small Lyapunov exponents). The existence of a phase transition is confirmed by a careful comparison of the spectra obtained with different approximations: deviations from linearity indeed become smaller for increasing order of the Markov process. Although it does not affect the entropy spectrum $g(\kappa)$, the phase transition is a further element pointing at the complexity of the underlying dynamics.

We conclude this study by stressing again that the absence of a global horseshoe mechanism makes the search for periodic orbits and the comprehension of the dynamics very difficult, since no algorithm exists which is able to give general answers. This is one of the manifestations of the uncomputability which underlies generalized shifts. Whether this is an oddity of this class of models or an observable property of more realistic systems with, for instance, a coupling between the x- and y-directions, remains an open question.

As a last remark, we comment on the fast convergence of the topological entropy as estimated with Markov processes. The performance of this approach is superior not only to the direct enumeration of the legal sequences, but also to the results inferred from the knowledge of the forbidden sequences (a method which yields an exponential convergence in the Hénon map [16]). The extension of this approach, unfortunately, is not straightforward since it requires construction of approximate Markov partitions in truly nonlinear systems.

We wish to thank P Grassberger, in the hope that he remembers the discussions we had long ago (Torino 1991), which led to a preliminary understanding of the model discussed herein.

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